Fountain Journal of Natural and Applied Sciences 2023; 12(2): 01-04

DOI: 10.53704/fujnas.v12i2.464
A publication of College of Natural and Applied Sciences, Fountain University, Osogbo, Nigeria.
Journal homepage: www.fountainjournals.com
ISSN: 2354-337X(Online),2350-1863(Print)

Extension of Short Rate Model Under a Lévy Process

Udoye, A. M.

Department of Mathematics, Federal University Oye-Ekiti, Ekiti

Abstract

A lot of abnormalities occur in real-life scenarios, thus leading to some difficulties in modelling such scenarios without deeper understanding of certain aspects of Lévy processes. In this paper, the short rate model of Hull-White (1990) is extended to a model for capturing possibilities of jumps in real-life situations using a class of Lévy processes called a variance gamma process.
Mathematics Subject Classification (2020). 91G30, 62P05

Keywords: Lévy processes, Brownian motion, Hull-White model, Variance gamma process

Introduction

Lévy processes was named after a French mathematician, Paul Lévy (1886-1971) who carried out a lot of research involving central limit theorem, law of large numbers, Gaussian processes, stable and infinitely divisible laws, and processes of independent and stationary increments. Other major contribution in this aspect between 1930s and 1940s include a Russian mathematician named Aleksandr Khintchine (1894-1959) who contributed in the fields of probability theory and number theory, and a Japanese mathematician named Kiyoshi Itô (1915-2008) who contributed in the aspect of random events. Lévy processes have become a popular tool for applications in physics, engineering, crude oil optionpricing, mathematical finance, etc. (Kyprianou, 2006; Papapantoleon, 2008; Swishchuk, 2008; Park *et al.*, 2014; Park & Kim, 2015; Udoye *et al*., 2021; Ma, 2003; Yang & Zang, 2001). Moreover, Rhee & Kim (2004) discussed market price of risk under the process. Madan & Seneta (1990) proposed a special type of the Lévy processes called a *variance gamma process* which they applied in share market returns, while Madan *et al*. (1998) discussed its application in option pricing. This class of Lévy process has been applied in engineering, science and finance by different researchers, Hirsa & Madan (2004) derived a partial integro-differential equation for Amerian options' pricing using the variance gamma process. Udoye & Ekhaguere (2022) applied the process in an interest rate derivative to capture jumps due to unexpected happenings, while Udoye & Akinola (2022) extended the work to involve special greeks of the extended model involving the process. Salem *et a*l. (2020) proposed modelling a water tank pump degradation driven by the process.

Chan *et al.* (1992) discussed empirical comparison of different models of interest rate driven by Brownian motion without consideration for the possibility of jumps due to unexpected

*Corresponding author:
Email address: princeadex26@gmail.com

happenings. Rachev *et al*. (2005) provided financial models for portfolio choosing, option pricing, credit and market risk control; they emphasized that some empirical evidence does not agree with the use of Gaussian distributions and α-stable distributions; while Kim *et al*. (2008) observed that the distribution of returns for assets possess weightier tails than the tails of the Gaussian distribution but thinner tails when compared to the α-stable distributions. Hence, the need to consider improved models with the ability to take care of special occurrences due to abnormality in real-life scenarios Various researchers have contributed to the applications of the Lévy processes in the field of mathematical finance (Udoye *et al*., 2021 & 2022; Klinger *et al.*, 2013). In this work, the Lévy process is applied in the extension of the interest rate model of Hull-White (1990) by considering the variance gamma process.

Methodology

The variance gamma process. Let X_t denote a variance gamma process. Then, Nicolleta (2021) provided a brief representation of the process as

$$X_t = wt + \theta G_t + \hat{\rho} W(G_t)$$

where w represents the cumulant generating function given by

$$w = \frac{1}{k} \ln\left(1 - \theta k - \frac{1}{2}\hat{\sigma}^2 k\right)$$

k is the variance of an arithmetic Brownian motion used in deriving the variance gamma process. $W(G_t)$ is given by

$$W(G_t) = \sqrt{G(t)} Z$$

such that

$$dX_t = wdt + \theta \Delta G(t) + \hat{\sigma} \Delta \sqrt{G(t)} Z$$

While

$$\Delta G(t) = G(t) - G(t_-).$$

Hull-White (1990) model: The model dynamics is given by

$$dr_t = (\beta - \alpha r_t) d_t + \sigma dW_t$$

where β, α σ and σ denote constant parameters for long-run mean rate, mean reversion's speed and volatility of the short rate, respectively. W_t denotes the Brownian motion. The solution to the dynamics is given by

$$r_t = r_0 \, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} d\, W_u.$$

Its mean is given by

$$E[r_t] = r_0 e^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t})$$

while as *t* tends to infinity, the mean becomes

$$E[r_t] = \frac{b}{a}.$$

Its variance is given by

$$Var[r_t] = \frac{\sigma^2}{2a}(1 - e^{-2\alpha t}).$$

Result

In this section, the Hull-White short rate model is extended from a process driven by a Brownian motion to a process with the ability to capture different jumps that occur due to unexpected happenings in real-life situations. Thus, the modified model will help in selecting a better model in financial markets (Lang *et al.*, 2018)

Let the short rate dynamics be given by

$$dr_t = (\beta - \alpha r_t) d_t + \sigma dX_t \qquad (3.1)$$

where X_t denotes the special class of Lévy process called a variance gamma process. Solving the dynamics gives

$$r_t = r_0 \, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} d\, X_u \qquad (3.2)$$

where $X_t = wt + \theta G_t + \hat{\sigma} W(G_t)$.

Here, $w = \frac{1}{k} \ln\left(1 - \theta k - \frac{1}{2}\hat{\sigma}^2 k\right)$

Solving equation (3.2) for $\int_0^t e^{\alpha u} d\, X_u$ gives

$$r_t = r_0\, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u} d\, X_u$$

$$= r_0\, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma \left(w \int_0^t e^{-\alpha(t-s)} ds + \sum_{s \in [0,t]} \theta \Delta G(s) e^{-\alpha(t-s)} + \sum_{s \in [0,t]} \hat{\sigma} \Delta \sqrt{G(s)} e^{-\alpha(t-s)} Z \right)$$

$$= r_0\, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma \left(w \int_0^t e^{-\alpha(t-s)} ds + \theta \sum_{s \in [0,t]} \Delta G(s) e^{-\alpha(t-s)} + \hat{\sigma} \sum_{s \in [0,t]} \Delta \sqrt{G(s)} e^{-\alpha(t-s)} Z \right)$$

Thus,

$$r_t = r_0\, \mathrm{e}^{-\alpha t} + \frac{b}{a}(1 - e^{-\alpha t}) + \sigma \left(\frac{w}{a}(1 - e^{-\alpha t}) + \theta \sum_{s \in [0,t]} \Delta G(s) e^{-\alpha(t-s)} + \hat{\sigma} \sum_{s \in [0,t]} \Delta \sqrt{G(s)} e^{-\alpha(t-s)} Z \right)$$

which gives the solution for the modified short rate model under a variance gamma process.

Conclusion

Lévy processes have contributed a lot in providing improved model that considers unexpected scenarios in real-life. A special class of the process called a variance gamma process has been used to obtain a modified Hull-White short rate model.

Conflict of Interest: There is no conflict of interest to be declared.

References

Chan, K. C., Karolyi, G. A., Longstaff, F. A. & Sanders, A. B. (1992). An empirical comparison of alternative models of the short-term interest rate. *The Journal of Finance*, 47(3), 1209-1227.

Hirsa, A. & Madan, D. B. (2004). Pricing American options under variance gamma. *Journal of Computational Finance* 7(2), 63-80.

Hull, J. & White, A. (1990). Pricing interest rate derivative securities. Review of Financial Studies 3, 573-592.

Klingler, S., Kim, Y. S., Rachev, S. T. & Fabozzi, F. J. (2013). Option pricing with time-changed Lévy Processes. *Applied Financial Economics* 23(15), 1231-1238.

Kyprianou, A. E. (2006). Introductory Lectures on Fluctuations of Lévy Processes with Applications. *Springer-Verlag Berlin Heidelberg*.

Lang, S., Signer, R. & Spremann, K. (2018). The choice of interest rate models and its effect on bank capital requirements regulation and financial stability. *International Journal of Economics and Finance* 10(1), 74-92.

Ma, C. (2003). Term structure of interest rates in the presence of Lévy jumps: The HJM approach. *Annals of Economics & Finance* 4, 401-426.

Madan, D.B., Carr, P. & Chang, E.C. (1998). The variance gamma process and option pricing. European Finance Review 2: 79-105

Madan, D.B. & Seneta, E. (1990). The variance gamma model for share market returns. *The Journal of Business* 63(4), 511-524

Papapantoleon, A. (2008). An Introduction to Lévy Processes with Applications in Finance.

Park, K. & Kim, S. (2015). On Interest Rate Option Pricing with Jump Processes. *International Journal of Engineering & Applied Sciences (IJEAS)* 2(7), 64-67.

Park, K., Kim, S. & Shaw, W.T. (2014). Estimation and simulation of bond option pricing on the arbitrage free model with jump. *Journal of Applied & Computational Mathematics* 3(2), 1-4.

Rachev, S. T., Menn, C. & Fabozzi, F. J. (2005). Fat-tailed and skewed asset return distributions: Implications for risk management, Portfolio

Selection and Option Pricing. John Wiley & Sons.

Rhee, J.H. & Kim, Y.T. (2004). The market price of risk on the Lévy Interest Rate Model. *Journal of Economic Research* 9, 1-28.

Salem, M. B., Fouladirad, M. & Deloux, E. (2020). Variance gamma process for predictive maintenance of mechanical systems, Cogr`es Lambda Mu 22 "Les risque au cœur des transitions." -22e Congr`e de Maˆıtrise des Risques, Oct 2020, Le Havres, France. hal-03295749v2.

Swishchuk, A. (2008). Multi-Factor Lévy Models for pricing financial and energy derivatives. *Canadian Applied Mathematics Quaterly* 17(44), 777-806.

Udoye, A. M. & Akinola, L. S. (2023). Special greeks of a variance-gamma driven vasicek model. *Scientific African*, 19, e01466.

Udoye, A. M.**,** Akinola, L. S., Annorzie, M. N. & Yakubu, Y. (2022). Sensitivity analysis of variance gamma parameters for interest rate derivatives. *IAENG International Journal of Applied Mathematics*, 52(2), 492-499.

Udoye, A. M. & G. O. S. Ekhaguere (2022). Sensitivity analysis of a class of interest rate derivatives in a variance gamma Lévy market. *Palestine Journal of Mathematics*. 11(2), 159-176.

Udoye, A. M.**,** C.P. Ogbogbo & L.S. Akinola (2021). Jump-diffusion process of interest Rates and the Malliavin calculus. *International Journal of Applied Mathematics*34(1),183-202. http://dx.doi.org/10.12732/ijam.v34i1.10.

Yang, H. & Zhang, L. (2001). Spectrally negative Lévy processes with applications in risk theory. *Advanced Applied Probability* 33, 281-291